Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 525–534
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

# Inequalities for convex sequences and their applications

Shanhe Wu [a], Lokenath Debnath [b,*]

[a] *Department of Mathematics, Longyan College, Longyan Fujian 364012, People's Republic of China*
[b] *Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539, USA*

Received 18 January 2007; accepted 20 February 2007

**Abstract**

Using the theory of majorization, new inequalities for convex sequences are proved. A necessary and sufficient condition for a convex sequence is established. This reveals an important relationship between the convex sequence and the majorized inequality. Finally, some applications of our new inequalities and some improvements of certain known results are presented.


*Keywords:* Convex sequence; Convex function; Majorization; Hermite–Hadamard's inequality; Maclaurin's inequality; Estimate

## 1. Introduction

Considerable attention has been given to the study of convex sequences and their properties, and the corresponding inequalities with applications. In general, convex sequences as discrete versions of convex functions play an important role in mathematical analysis and in the theory of inequalities. Inequalities for convex sequences provided considerable interest in proving a large number of elegant results with applications (see Wu and Shi [1] and Mercer [2]). In addition, several authors including Mitrinović and Vasić [3], Roberts and Varberg [4], and Mitrinović et al. [5] presented a large number of major results for convex sequences and related inequalities.

The major objective of this paper is to prove some new inequalities for convex sequences with applications. Using analytic techniques and the theory of majorization (see Marshall and Olkin [6] and Wu [7,8]), a necessary and sufficient condition for a convex sequence is proved. An important relationship between convex sequences and majorized inequalities is obtained. This result can be effectively used to discover and prove new inequalities for convex sequences. Moreover, many useful inequalities related to convex functions as well as convex sequences are established. Some of these inequalities provide a refined estimation for the sum of certain finite sequences. In addition, examples of applications of the obtained results are presented in the last section of this paper.

We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers respectively. $\mathcal{I}$ denotes interval. Let $x = (x_1, x_2, \ldots, x_n)$ denote $n$-tuple ($n$-dimensional real vector). We denote the vector space $\mathbb{R}^n$ by

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}.$$

* Corresponding author. Tel.: +1 956 381 3459; fax: +1 956 384 5091.
*E-mail addresses:* wushanhe@yahoo.com.cn (S. Wu), debnathl@utpa.edu (L. Debnath).

0898-1221/$ - see front matter 
doi:10.1016/j.camwa.2007.02.005

**Definition 1** (*Mitrinović and Vasić [3], p. 11*). A real sequence $\{a_i\}_{i=1}^n$ is said to be a convex sequence if

$$2a_i \leqslant a_{i-1} + a_{i+1} \quad \text{for all } i = 2, 3, \ldots, n-1. \tag{1}$$

A real sequence $\{a_i\}_{i=1}^n$ is said to be a concave sequence if the reverse inequality in (1) holds.

**Definition 2** (*Marshall and Olkin [6], p. 7*). For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, let $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant y_{[2]} \geqslant \cdots \geqslant y_{[n]}$ denote the components of $x$ and $y$ in decreasing order, respectively. The $n$-tuple $y$ is said to majorize $x$ (or $x$ is to be majorized by $y$) in symbols $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \quad \text{holds for } k = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{2}$$

**Definition 3** (*Marshall and Olkin [6], p. 78*). Let $x \in \mathbb{R}^n$, we define the $k$th symmetric function as follows

$$\sigma_k(x) = \sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} x_{i_j}, \quad k = 1, 2, \ldots, n. \tag{3}$$

## 2. Lemmas

To prove the results in Sections 3 and 4, we need the following lemmas.

**Lemma 1** (*Marshall and Olkin [6], p. 11*). *Let $(x_1, x_2, \ldots, x_n) \prec (y_1, y_2, \ldots, y_n)$, $x_i, y_i \in \mathcal{I}$, $i = 1, 2, \ldots n$. Then for any continuous convex function $\phi : \mathcal{I} \to \mathbb{R}$, the following inequality holds*

$$\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n) \leqslant \phi(y_1) + \phi(y_2) + \cdots + \phi(y_n). \tag{4}$$

*For any continuous concave function $\phi : \mathcal{I} \to \mathbb{R}$, the reverse inequality of (4) holds.*

**Lemma 2** (*Hermite–Hadamard's Inequality (See Mitrinović et al. [5], p. 10)*). *Let $f$ be a convex function on $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \leqslant \frac{f(a) + f(b)}{2}, \tag{5}$$

*with equality holding if and only if $f$ is a linear function.*

*If $f$ is a concave function on $[a, b]$, then two inequalities in (5) are reversed.*

**Lemma 3** (*Maclaurin's Inequality (See Hardy et al. [9], p. 52)*). *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \in \mathbb{N}$. Then*

$$\left[\frac{\sigma_n(x_1, x_2, \ldots, x_n)}{\binom{n}{n}}\right]^{\frac{1}{n}} \leqslant \cdots \leqslant \left[\frac{\sigma_2(x_1, x_2, \ldots, x_n)}{\binom{n}{2}}\right]^{\frac{1}{2}} \leqslant \left[\frac{\sigma_1(x_1, x_2, \ldots, x_n)}{\binom{n}{1}}\right]. \tag{6}$$

**Lemma 4.** *Let $x, y \in \mathbb{R}^n$, $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$, $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If there exists $m (1 \leqslant m \leqslant n, m \in \mathbb{N})$ such that $x_i \leqslant y_i$ for $i = 1, 2, \ldots, m$, and $x_i \geqslant y_i$ for $i = m+1, m+2, \ldots, n$. Then $x \prec y$.*

**Proof.** It follows from the hypotheses of Lemma 3 that, for $1 \leqslant k \leqslant m$, we have

$$\sum_{i=1}^{k} x_{[i]} = \sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i \leqslant \sum_{i=1}^{k} y_{[i]}; \tag{7}$$

for $m+1 \leqslant k \leqslant n-1$, we have

$$\sum_{i=1}^{k} x_{[i]} = \sum_{i=1}^{k} x_i = \sum_{i=1}^{n} x_i - \sum_{i=k+1}^{n} x_i \leqslant \sum_{i=1}^{n} x_i - \sum_{i=k+1}^{n} y_i \leqslant \sum_{i=1}^{n} x_i - \sum_{i=k+1}^{n} y_{[i]}$$
$$= \sum_{i=1}^{n} y_{[i]} - \sum_{i=k+1}^{n} y_{[i]} = \sum_{i=1}^{k} y_{[i]}. \tag{8}$$

Combining (7) and (8) leads to

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n-1. \tag{9}$$

Using the condition $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we conclude from the Definition 2 that $x \prec y$. □

**Lemma 5.** *Let $\{a_i\}_{i=1}^{n}$ be a convex sequence, and let $A_k = \frac{1}{k}\sum_{i=1}^{k} a_i$. Then $\{A_k\}_{k=1}^{n}$ also is a convex sequence.*

**Proof.** Define

$$f(k) = k(k+1)(k-1)(2A_k - A_{k+1} - A_{k-1}), \quad k = 2, 3, \dots, n-1.$$

We deduce from the hypotheses in Lemma 5 that

$$\begin{aligned} f(k) - f(k-1) &= k(k+1)(k-1)(2A_k - A_{k+1} - A_{k-1}) - (k-1)k(k-2)(2A_{k-1} - A_k - A_{k-2}) \\ &= 2(k-1)(k+1)\sum_{i=1}^{k} a_i - k(k-1)\sum_{i=1}^{k+1} a_i - k(k+1)\sum_{i=1}^{k-1} a_i \\ &\quad - 2k(k-2)\sum_{i=1}^{k-1} a_i + (k-1)(k-2)\sum_{i=1}^{k} a_i + k(k-1)\sum_{i=1}^{k-2} a_i \\ &= k(k-1)(2a_k - a_{k+1} - a_{k-1}) \leqslant 0, \end{aligned}$$

that is,

$$f(k) \leqslant f(k-1) \quad \text{for } k = 3, 4, \dots, n-1.$$

Thus,

$$f(k) \leqslant f(k-1) \leqslant \cdots \leqslant f(2) = 6(2a_2 - a_3 - a_1) \leqslant 0,$$

which yields

$$2A_k \leqslant A_{k+1} + A_{k-1} \quad \text{for } k = 2, 3, \dots, n-1.$$

Consequently $\{A_k\}_{k=1}^{n}$ is a convex sequence. □

**Lemma 6.** *Let $x_1, x_2, \dots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i \leqslant 1$. Then the sequences $\{\sigma_k(x_1, x_2, \dots, x_n)\}_{k=1}^{n}$ and $\{\ln(1/\sigma_k(x_1, x_2, \dots, x_n))\}_{k=1}^{n}$ are convex.*

**Proof.** Applying the Maclaurin inequality (see Lemma 3):

$$\left[\frac{\sigma_k(x_1, x_2, \dots, x_n)}{\binom{n}{k}}\right]^{\frac{1}{k}} \leqslant \left[\frac{\sigma_{k-1}(x_1, x_2, \dots, x_n)}{\binom{n}{k-1}}\right]^{\frac{1}{k-1}} \leqslant \cdots \leqslant \left[\frac{\sigma_1(x_1, x_2, \dots, x_n)}{\binom{n}{1}}\right],$$

together with the condition $\sigma_1(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i \leqslant 1$, it follows that, for $k = 2, 3, \dots, n-1$,

$$2\sigma_k(x_1, x_2, \dots, x_n) \leqslant \frac{2\binom{n}{k}}{\binom{n}{k-1}}\left[\frac{\sigma_{k-1}(x_1, x_2, \dots, x_n)}{\binom{n}{k-1}}\right]^{\frac{1}{k-1}} \sigma_{k-1}(x_1, x_2, \dots, x_n)$$

$$\leqslant \frac{2\binom{n}{k}}{\binom{n}{k-1}}\left[\frac{\sigma_1(x_1,x_2,\ldots,x_n)}{\binom{n}{1}}\right]\sigma_{k-1}(x_1,x_2,\ldots,x_n)$$
$$\leqslant \frac{2\binom{n}{k}}{\binom{n}{k-1}\binom{n}{1}}\sigma_{k-1}(x_1,x_2,\ldots,x_n) = \frac{2}{k}\left(1-\frac{k-1}{n}\right)\sigma_{k-1}(x_1,x_2,\ldots,x_n)$$
$$< \sigma_{k-1}(x_1,x_2,\ldots,x_n) < \sigma_{k-1}(x_1,x_2,\ldots,x_n)+\sigma_{k+1}(x_1,x_2,\ldots,x_n).$$

Hence, we conclude that $\{\sigma_k(x_1,x_2,\ldots,x_n)\}_{k=1}^n$ is a convex sequence.

In addition, we deduce that the sequence $\{\ln(1/\sigma_k(x_1,x_2,\ldots,x_n))\}_{k=1}^n$ is convex from the known result of Wu [10]:

$$(1/\sigma_k(x_1,x_2,\ldots,x_n))^2 \leqslant (1/\sigma_{k-1}(x_1,x_2,\ldots,x_n))\,(1/\sigma_{k+1}(x_1,x_2,\ldots,x_n))\,. \tag{10}$$

Or, equivalently,

$$2\ln(1/\sigma_k(x_1,x_2,\ldots,x_n)) \leqslant \ln(1/\sigma_{k-1}(x_1,x_2,\ldots,x_n)) + \ln(1/\sigma_{k+1}(x_1,x_2,\ldots,x_n))\,.$$

Thus Lemma 6 is proved. □

**Lemma 7.** *Let $\{a_i\}_{i=1}^n$ be a convex sequence, let $\psi$ be a continuous convex function and increasing on $\Omega$, and let the function $\phi : [1,n] \to \mathcal{I}(\mathcal{I} \subseteq \Omega)$ defined by*

$$\phi(x) = \begin{cases} a_1 + (a_2-a_1)(x-1), & 1 \leqslant x < 2 \\ a_2 + (a_3-a_2)(x-2), & 2 \leqslant x < 3 \\ \cdots & \\ a_i + (a_{i+1}-a_i)(x-i), & i \leqslant x < i+1 \\ \cdots & \\ a_{n-1} + (a_n-a_{n-1})(x-n+1), & n-1 \leqslant x \leqslant n. \end{cases} \tag{11}$$

*Then $\psi(\phi(x))$ is a continuous and convex function on $[1,n]$.*

*If $\psi$ is a continuous concave function and decreasing on $\Omega$. Then $\psi(\phi(x))$ is a continuous and concave function on $[1,n]$.*

**Proof.** It is easy to see that $\psi(\phi(x))$ is continuous on $[1,n]$ from the definition of $\psi(\phi(x))$. Now we prove that $\psi(\phi(x))$ is convex on $[1,n]$.

Consider two arbitrary real numbers $x_1, x_2 \in [1,n]$ (without loss of generality we assume that $x_1 < x_2$). Then there exists positive integers $k, l, m$ $(1 \leqslant k \leqslant l \leqslant m \leqslant n-1)$ such that

$$x_1 \in [k,k+1), \quad (x_1+x_2)/2 \in [l,l+1), \quad x_2 \in [m,m+1].$$

Further, it follows from (11) that

$$\phi(x_1) = a_k + (a_{k+1}-a_k)(x_1-k),$$
$$\phi(x_2) = a_m + (a_{m+1}-a_m)(x_2-m),$$
$$\phi\left(\frac{x_1+x_2}{2}\right) = a_l + (a_{l+1}-a_l)\left(\frac{x_1+x_2}{2}-l\right).$$

On the other hand, by the definition of a convex sequence together with the hypotheses $k \leqslant x_1 < k+1$, $k \leqslant l$, it follows that

$$\begin{aligned}\phi(x_1) &= a_k + (a_{k+1}-a_k)(x_1-k) \\ &= a_{k+1} + (a_{k+2}-a_{k+1})(x_1-k-1) + (2a_{k+1}-a_{k+2}-a_k)(x_1-k-1) \\ &\geqslant a_{k+1} + (a_{k+2}-a_{k+1})(x_1-k-1) \geqslant \cdots \geqslant a_l + (a_{l+1}-a_l)(x_1-l).\end{aligned} \tag{12}$$

Similarly, from $m \leqslant x_2 \leqslant m+1$ and $m \geqslant l$, we obtain

$$\begin{aligned}\phi(x_2) &= a_m + (a_{m+1} - a_m)(x_2 - m) \\ &= a_{m-1} + (a_m - a_{m-1})(x_2 - m + 1) + (a_{m+1} + a_{m-1} - 2a_m)(x_2 - m) \\ &\geqslant a_{m-1} + (a_m - a_{m-1})(x_2 - m + 1) \geqslant \cdots \geqslant a_l + (a_{l+1} - a_l)(x_2 - l).\end{aligned} \tag{13}$$

Combining (12) and (13) yields

$$\begin{aligned}\frac{\phi(x_1)+\phi(x_2)}{2} &\geqslant \frac{a_l + (a_{l+1} - a_l)(x_1 - l) + a_l + (a_{l+1} - a_l)(x_2 - l)}{2} \\ &= a_l + (a_{l+1} - a_l)\left(\frac{x_1+x_2}{2} - l\right) = \phi\left(\frac{x_1+x_2}{2}\right).\end{aligned}$$

Now from the fact that $\psi$ is convex and increasing function, we obtain

$$\psi\left(\phi\left(\frac{x_1+x_2}{2}\right)\right) \leqslant \psi\left(\frac{\phi(x_1)+\phi(x_2)}{2}\right) \leqslant \frac{\psi(\phi(x_1)) + \psi(\phi(x_2))}{2}.$$

Hence, we conclude that $\psi(\phi(x))$ is convex on $[1, n]$.

The second part of Lemma 7 can be proved in a similar way. The proof of Lemma 7 is thus complete. □

## 3. Main results

In this section, we state the main results.

**Theorem 1.** *Let $\{a_i\}_{i=1}^n$ be a convex sequence, $a_1, a_2, \dots, a_n \in \mathcal{I}$, and let $\psi$ be a continuous convex function and increasing on $\mathcal{I}$. Then for any $(p_1, p_2, \dots, p_k) \prec (q_1, q_2, \dots, q_k)$ $(1 \leqslant p_i \leqslant n, 1 \leqslant q_i \leqslant n, p_i, q_i \in \mathbb{N}, i = 1, 2, \dots, k, k \geqslant 2)$, the following inequality holds*

$$\psi(a_{p_1}) + \psi(a_{p_2}) + \cdots + \psi(a_{p_k}) \leqslant \psi(a_{q_1}) + \psi(a_{q_2}) + \cdots + \psi(a_{q_k}). \tag{14}$$

*If $\psi$ is a continuous concave function and decreasing on $\mathcal{I}$, then the inequality (14) is reversed.*

**Proof.** Define a function $\phi(x)$ as (11). We conclude from Lemma 7 that $\psi(\phi(x))$ is continuous and convex on $[1, n]$.

Using the Lemma 1 together with $(p_1, p_2, \dots, p_k) \prec (q_1, q_2, \dots, q_k)$ $(1 \leqslant p_i \leqslant n, 1 \leqslant q_i \leqslant n)$ yields

$$\psi(\phi(p_1)) + \psi(\phi(p_2)) + \cdots + \psi(\phi(p_k)) \leqslant \psi(\phi(q_1)) + \psi(\phi(q_2)) + \cdots + \psi(\phi(q_k)).$$

Now, it follows at once from $\phi(i) = a_i$, $i = 1, 2, \dots, n$ that the desired inequality

$$\psi(a_{p_1}) + \psi(a_{p_2}) + \cdots + \psi(a_{p_k}) \leqslant \psi(a_{q_1}) + \psi(a_{q_2}) + \cdots + \psi(a_{q_k}).$$

Similarly to the above, the second part of Theorem 1 can be proved. The proof is complete. □

We choose that $\psi(x) = x^\lambda$ $(\lambda \geqslant 1)$ in (14). Clearly, $\psi$ is convex and increasing on $[0, +\infty)$. It follows from Theorem 1 that

**Corollary 1.** *Let $\{a_i\}_{i=1}^n$ be a nonnegative convex sequence, and let $\lambda \geqslant 1$, $\lambda \in \mathbb{R}$, $(p_1, p_2, \dots, p_k) \prec (q_1, q_2, \dots, q_k)$, $1 \leqslant p_i \leqslant n$, $1 \leqslant q_i \leqslant n$, $p_i, q_i \in \mathbb{N}$, $i = 1, 2, \dots, k$, $k \geqslant 2$. Then*

$$(a_{p_1})^\lambda + (a_{p_2})^\lambda + \cdots + (a_{p_k})^\lambda \leqslant (a_{q_1})^\lambda + (a_{q_2})^\lambda + \cdots + (a_{q_k})^\lambda. \tag{15}$$

**Theorem 2.** *A necessary and sufficient condition that $\{a_i\}_{i=1}^n$ is a convex sequence is that the inequality*

$$a_{p_1} + a_{p_2} + \cdots + a_{p_k} \leqslant a_{q_1} + a_{q_2} + \cdots + a_{q_k} \tag{16}$$

*holds for any $(p_1, p_2, \dots, p_k) \prec (q_1, q_2, \dots, q_k)$, $1 \leqslant p_i \leqslant n$, $1 \leqslant q_i \leqslant n$, $p_i, q_i \in \mathbb{N}$, $i = 1, 2, \dots, k$, $k \geqslant 2$.*

**Proof.** The necessity part follows at once from Theorem 1 with $\psi(x) = x$ $(x \in \mathbb{R})$. On the other hand, by the obvious relationship for majorization: $(i, i) \prec (i+1, i-1)$, $i = 2, 3, \dots, n-1$, we deduce from (16) that $2a_i \leqslant a_{i-1} + a_{i+1}$ for all $i = 2, 3, \dots, n-1$. Consequently, the sequence $\{a_i\}_{i=1}^n$ is convex. The sufficient condition is proved. □

**Theorem 3.** *Let $\{a_i\}_{i=1}^n$ be a positive convex sequence and strictly increasing, and let $f$ be a convex and nonnegative function on $(0, +\infty)$. Then*

$$f(a_1) + f(a_n) + \frac{1}{a_2 - a_1} \int_{(a_1+a_2)/2}^{(3a_{n-1}-a_{n-2})/2} f(x)\mathrm{d}x \geqslant \sum_{i=1}^{n} f(a_i)$$
$$\geqslant \frac{1}{2}\left(f(a_1) + f(a_n)\right) + \frac{1}{a_n - a_{n-1}} \int_{a_1}^{a_n} f(x)\mathrm{d}x \tag{17}$$

*with equality holding if and only if $f$ is a linear function, and $a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$.*

*Suppose that $\{a_i\}_{i=1}^n$ is a positive concave sequence and strictly increasing, and $f$ is concave and nonnegative on $(0, +\infty)$. Then two inequalities in (17) are reversed.*

**Proof.** It follows from the hypotheses and the Hermite–Hadamard inequality (see Lemma 2) that

$$\frac{f(a_i) + f(a_{i+1})}{2} \geqslant \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} f(x)\mathrm{d}x,$$

and

$$0 < a_{i+1} - a_i \leqslant a_{i+2} - a_{i+1} \leqslant \cdots \leqslant a_n - a_{n-1} \quad (i = 1, 2, \ldots n-1).$$

Hence, we have

$$\begin{aligned}
\sum_{i=1}^{n} f(a_i) &= \frac{1}{2}\left(f(a_1) + f(a_n)\right) + \sum_{i=1}^{n-1} \frac{f(a_i) + f(a_{i+1})}{2} \\
&\geqslant \frac{1}{2}\left(f(a_1) + f(a_n)\right) + \sum_{i=1}^{n-1} \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} f(x)\mathrm{d}x \\
&\geqslant \frac{1}{2}\left(f(a_1) + f(a_n)\right) + \frac{1}{a_n - a_{n-1}} \sum_{i=1}^{n-1} \int_{a_i}^{a_{i+1}} f(x)\mathrm{d}x \\
&= \frac{1}{2}\left(f(a_1) + f(a_n)\right) + \frac{1}{a_n - a_{n-1}} \int_{a_1}^{a_n} f(x)\mathrm{d}x.
\end{aligned}$$

On the other hand, by using the left-hand side of Hermite–Hadamard's inequality, we have

$$f(a_i) = f\left(\frac{(a_{i-1} + a_i)/2 + (3a_i - a_{i-1})/2}{2}\right) \leqslant \frac{1}{a_i - a_{i-1}} \int_{(a_{i-1}+a_i)/2}^{(3a_i-a_{i-1})/2} f(x)\mathrm{d}x.$$

Consequently, we deduce from $a_{i+1} - a_i \geqslant a_i - a_{i-1} \geqslant \cdots \geqslant a_2 - a_1 > 0$ $(i = 1, 2, \ldots n-1)$ that

$$f(a_{n-1}) \leqslant \frac{1}{a_2 - a_1} \int_{(a_{n-2}+a_{n-1})/2}^{(3a_{n-1}-a_{n-2})/2} f(x)\mathrm{d}x,$$

and

$$f(a_i) \leqslant \frac{1}{a_2 - a_1} \int_{(a_{i-1}+a_i)/2}^{(a_i+a_{i+1})/2} f(x)\mathrm{d}x \quad \text{for } i = 2, 3, \ldots n-2.$$

(The latter inequality follows from $f(x) \geqslant 0$ and $(3a_i - a_{i-1})/2 \leqslant (a_i + a_{i+1})/2$).

Utilizing the above inequalities, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} f(a_i) &= f(a_1) + f(a_n) + f(a_{n-1}) + \sum_{i=2}^{n-2} f(a_i) \\
&\leqslant f(a_1) + f(a_n) + \frac{1}{a_2 - a_1} \int_{(a_{n-2}+a_{n-1})/2}^{(3a_{n-1}-a_{n-2})/2} f(x)\mathrm{d}x + \frac{1}{a_2 - a_1} \sum_{i=2}^{n-2} \int_{(a_{i-1}+a_i)/2}^{(a_i+a_{i+1})/2} f(x)\mathrm{d}x \\
&= f(a_1) + f(a_n) + \frac{1}{a_2 - a_1} \int_{(a_1+a_2)/2}^{(3a_{n-1}-a_{n-2})/2} f(x)\mathrm{d}x.
\end{aligned}$$

Similarly, we can prove the second part of Theorem 3. The proof is complete. □

**Remark 1.** For any arithmetic progression $\{a+kd\}_{k=1}^{n}$, we find $2a_k = a_{k-1}+a_{k+1}$ $(k = 2, 3, \ldots, n-1)$. This means that the arithmetic progression $\{a + kd\}_{k=1}^{n}$ is convex as well as concave. Consequently, by using of Theorem 3, we obtain

**Corollary 2.** *Let $a \geqslant 0$, $d > 0$, $n \geqslant 3$, and let $f$ be a convex and nonnegative function on $(0, +\infty)$. Then*

$$f(a+d) + f(a+nd) + \frac{1}{d}\int_{(2a+3d)/2}^{(2a+2nd-d)/2} f(x)\mathrm{d}x \geqslant \sum_{k=1}^{n} f(a+kd)$$
$$\geqslant \frac{1}{2}\left(f(a+d) + f(a+nd)\right) + \frac{1}{d}\int_{a+d}^{a+nd} f(x)\mathrm{d}x \quad (18)$$

*with equality holding if and only if $f$ is a linear function.*

*If $f$ is a concave and nonnegative function on $(0, +\infty)$, then two inequalities in (18) are reversed.*

## 4. Some applications

In this section, we show some applications of our new inequalities. Moreover, improvement of certain known results is also presented.

**Proposition 1.** *Let $\{a_i\}_{i=1}^{n}$ be a nonnegative convex sequence, $\lambda \geqslant 1$, $\lambda \in \mathbb{R}$, $m, n, l \in \mathbb{N}$. Then*

$$\frac{m}{l^{\lambda}}\left(\sum_{i=1}^{l} a_i\right)^{\lambda} + \frac{n-m-l}{n^{\lambda}}\left(\sum_{i=1}^{n} a_i\right)^{\lambda} \geqslant \frac{n-l}{(n-m)^{\lambda}}\left(\sum_{i=1}^{n-m} a_i\right)^{\lambda} \quad (n > m + l), \quad (19)$$

$$\frac{n-m}{l^{\lambda}}\left(\sum_{i=1}^{l} a_i\right)^{\lambda} + \frac{m-l}{n^{\lambda}}\left(\sum_{i=1}^{n} a_i\right)^{\lambda} \geqslant \frac{n-l}{m^{\lambda}}\left(\sum_{i=1}^{m} a_i\right)^{\lambda} \quad (n > m > l). \quad (20)$$

**Proof.** Since $\{a_i\}_{i=1}^{n}$ is a convex sequence, it follows from Lemma 5 that $\{A_k\}_{k=1}^{n}$ also is a convex sequence, where $A_k = \frac{1}{k}\sum_{i=1}^{k} a_i$.

On the other hand, utilizing Lemma 4, it is easy to verify that

$$\Big(\underbrace{n, \ldots, n}_{m+l}, \underbrace{n-m, \ldots, n-m}_{n-l}\Big) \prec \Big(\underbrace{n, \ldots, n}_{n}, \underbrace{l, \ldots, l}_{m}\Big) \quad (n > m + l), \quad (21)$$

$$\Big(\underbrace{m, \ldots, m}_{n-l}\Big) \prec \Big(\underbrace{n, \ldots, n}_{m-l}, \underbrace{l, \ldots, l}_{n-m}\Big) \quad (n > m > l). \quad (22)$$

Hence, by Corollary 1, we obtain

$$(m + l)A_n^{\lambda} + (n - l)A_{n-m}^{\lambda} \leqslant nA_n^{\lambda} + mA_l^{\lambda} \quad (n > m + l), \quad (23)$$

$$(n - l)A_m^{\lambda} \leqslant (m - l)A_n^{\lambda} + (n - m)A_l^{\lambda} \quad (n > m > l). \quad (24)$$

The above inequalities can be translated to inequalities (19) and (20) respectively, since $A_k = \frac{1}{k}\sum_{i=1}^{k} a_i$, $k = 1, 2, \ldots, n$.

Mercer [11] established the following inequality for the convex sequence $\{u_k\}_{k=1}^{n}$

$$\sum_{k=1}^{n}\left[\frac{1}{n} - \frac{1}{2^{n-1}}\binom{n-1}{k-1}\right]u_k \geqslant 0. \quad (25)$$

In a recent paper, Wu [12] proved the following relationship for majorization:

$$\left(\underbrace{m+nl,\ldots,m+nl}_{nC_{n-1}^{n-1}},\ldots,\underbrace{m+2l,\ldots,m+2l}_{nC_{n-1}^{1}},\underbrace{m+l,\ldots,m+l}_{nC_{n-1}^{0}}\right)$$
$$\prec\left(\underbrace{m+nl,\ldots,m+nl}_{2^{n-1}},\ldots,\underbrace{m+2l,\ldots,m+2l}_{2^{n-1}},\underbrace{m+l,\ldots,m+l}_{2^{n-1}}\right),\tag{26}$$

where $m \geqslant 0, n \geqslant 3, l \geqslant 1, m, n, l \in \mathbb{N}$. □

Utilizing the Corollary 1, Theorem 2 and (26), we can generalize Mercer's inequality (25) as follows:

**Proposition 2.** *Let $m \geqslant 0$, $n \geqslant 3$, $l \geqslant 1$, $m, n, l \in \mathbb{N}$, $\lambda \geqslant 1$, $\lambda \in \mathbb{R}$. If $\{u_k\}_{k=1}^{m+nl}$ is a convex sequence, then*

$$\sum_{k=1}^{n}\left[\frac{1}{n}-\frac{1}{2^{n-1}}\binom{n-1}{k-1}\right]u_{m+kl}\geqslant 0.\tag{27}$$

*If $\{u_k\}_{k=1}^{m+nl}$ is a nonnegative convex sequence, then*

$$\sum_{k=1}^{n}\left[\frac{1}{n}-\frac{1}{2^{n-1}}\binom{n-1}{k-1}\right]u_{m+kl}^{\lambda}\geqslant 0.\tag{28}$$

In the following proposition, we establish a class of new integral inequalities.

**Proposition 3.** *Let $f_1, f_2, \ldots, f_m$ be positive and integrable functions on $[a, b]$, $n \geqslant 3$, $m, n \in \mathbb{N}$. Then*

$$\left[\prod_{k=2}^{n-1}\left(\sum_{i=1}^{m}\int_a^b f_i^k(x)\mathrm{d}x\right)\right]^{\frac{n}{n-2}}\leqslant\prod_{k=1}^{n}\left(\sum_{i=1}^{m}\int_a^b f_i^k(x)\mathrm{d}x\right)$$
$$\leqslant\left[\left(\sum_{i=1}^{m}\int_a^b f_i(x)\mathrm{d}x\right)\left(\sum_{i=1}^{m}\int_a^b f_i^n(x)\mathrm{d}x\right)\right]^{\frac{n}{2}}.\tag{29}$$

**Proof.** We define a sequence $\{I_k\}_{k=1}^{n}$ by $I_k=\ln\left(\sum_{i=1}^{m}\int_a^b f_i^k(x)\mathrm{d}x\right)$.

Applying the well-known Buniakowski–Cauchy–Schwarz inequality (Mitrinović and Vasić [3], p. 43), it follows that

$$\ln\left(\sum_{i=1}^{m}\int_a^b f_i^{k-1}(x)\mathrm{d}x\right)+\ln\left(\sum_{i=1}^{m}\int_a^b f_i^{k+1}(x)\mathrm{d}x\right)=\ln\left[\left(\sum_{i=1}^{m}\int_a^b f_i^{k-1}(x)\mathrm{d}x\right)\left(\sum_{i=1}^{m}\int_a^b f_i^{k+1}(x)\mathrm{d}x\right)\right]$$
$$=\ln\left[\left(\int_a^b\left(\sum_{i=1}^{m}f_i^{k-1}(x)\right)\mathrm{d}x\right)\left(\int_a^b\left(\sum_{i=1}^{m}f_i^{k+1}(x)\right)\mathrm{d}x\right)\right]$$
$$\geqslant\ln\left(\int_a^b\sqrt{\left(\sum_{i=1}^{m}f_i^{k-1}(x)\right)\left(\sum_{i=1}^{m}f_i^{k+1}(x)\right)}\mathrm{d}x\right)^2$$
$$\geqslant\ln\left(\int_a^b\sum_{i=1}^{m}f_i^{k}(x)\mathrm{d}x\right)^2=2\ln\left(\sum_{i=1}^{m}\int_a^b f_i^{k}(x)\mathrm{d}x\right),$$

which implies $I_{k-1}+I_{k+1}\geqslant 2I_k$, $k=2,3,\ldots,n-1$. It follows that $\{I_k\}_{k=1}^{n}$ is a convex sequence.

On the other hand, it is easy to verify by Lemma 4 that

$$\left(\underbrace{n-1,\ldots,n-1}_{n},\underbrace{n-2,\ldots,n-2}_{n},\ldots,\underbrace{3,\ldots,3}_{n},\underbrace{2,\ldots,2}_{n}\right)$$
$$\prec\left(\underbrace{n,\ldots,n}_{n-2},\underbrace{n-1,\ldots,n-1}_{n-2},\ldots,\underbrace{2,\ldots,2}_{n-2},\underbrace{1,\ldots,1}_{n-2}\right),\tag{30}$$

$$(n,n,n-1,n-1,\ldots,2,2,1,1)\prec\left(\underbrace{n,\ldots,n}_{n},\underbrace{1,\ldots,1}_{n}\right).\tag{31}$$

Hence, it follows from Theorem 2 that

$$n(I_{n-1}+I_{n-2}+\cdots+I_3+I_2)\leqslant(n-2)(I_n+I_{n-1}+\cdots+I_2+I_1),$$
$$2(I_n+I_{n-1}+\cdots+I_2+I_1)\leqslant n(I_n+I_1).$$

The inequality (29) follows directly from the above inequalities. The proof is complete. □

**Remark 2.** A special case of $m=1$ in (29) yields the inequalities:

$$\left(\left(\int_a^b f^2(x)\mathrm{d}x\right)\left(\int_a^b f^3(x)\mathrm{d}x\right)\cdots\left(\int_a^b f^{n-1}(x)\mathrm{d}x\right)\right)^{\frac{n}{n-2}}$$
$$\leqslant\left(\int_a^b f(x)\mathrm{d}x\right)\left(\int_a^b f^2(x)\mathrm{d}x\right)\cdots\left(\int_a^b f^n(x)\mathrm{d}x\right)\leqslant\left(\left(\int_a^b f(x)\mathrm{d}x\right)\left(\int_a^b f^n(x)\mathrm{d}x\right)\right)^{\frac{n}{2}},\tag{32}$$

where $f$ is a positive and integrable function on $[a,b]$, $n\geqslant 3$, $n\in\mathbb{N}$.

**Proposition 4.** *Let $x_1,x_2,\ldots,x_n$ be positive real numbers with $\sum_{i=1}^n x_i\leqslant 1$, and let $n-l\geqslant m>l$, $m,n,l\in\mathbb{N}$. Then*

$$\frac{m}{m-l}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^{m}x_{i_j}\right)+\frac{l}{l-m}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^{l}x_{i_j}\right)\leqslant\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^{m+l}x_{i_j}$$
$$\leqslant\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^{m}x_{i_j}\right)^{\frac{m}{m-l}}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\prod_{j=1}^{l}x_{i_j}\right)^{\frac{l}{l-m}}.\tag{33}$$

**Proof.** By Lemma 4, we conclude that

$$\left(\underbrace{m,\ldots,m}_{m}\right)\prec\left(\underbrace{m+l,\ldots,m+l}_{m-l},\underbrace{l,\ldots,l}_{l}\right)\quad(m>l).\tag{34}$$

Now, using Lemma 6 and Theorem 2, we obtain

$$m\sigma_m(x_1,x_2,\ldots,x_n)\leqslant(m-l)\sigma_{m+l}(x_1,x_2,\ldots,x_n)+l\sigma_l(x_1,x_2,\ldots,x_n),$$
$$m\ln(1/\sigma_m(x_1,x_2,\ldots,x_n))\leqslant(m-l)\ln(1/\sigma_{m+l}(x_1,x_2,\ldots,x_n))+l\ln(1/\sigma_l(x_1,x_2,\ldots,x_n)).$$

The inequality (33) follows immediately from the above inequalities.

Finally, we show an important application of Corollary 2. A simple and effective method for estimating the sum of certain finite sequence is given via the following proposition. □

**Proposition 5.** *Let $a \geqslant 0$, $d > 0$, $n \geqslant 3$. Then for $\lambda < 0$ or $\lambda > 1$, we have the inequalities*

$$(a+d)^\lambda + (a+nd)^\lambda + \frac{1}{d(\lambda+1)}\left(\left(a+nd-\frac{1}{2}d\right)^{\lambda+1} - \left(a+\frac{3}{2}d\right)^{\lambda+1}\right) > \sum_{k=1}^{n}(a+kd)^\lambda$$
$$> \frac{(a+d)^\lambda + (a+nd)^\lambda}{2} + \frac{1}{d(\lambda+1)}\left((a+nd)^{\lambda+1} - (a+d)^{\lambda+1}\right). \tag{35}$$

*For $0 < \lambda < 1$, we have the inequalities*

$$(a+d)^\lambda + (a+nd)^\lambda + \frac{1}{d(\lambda+1)}\left(\left(a+nd-\frac{1}{2}d\right)^{\lambda+1} - \left(a+\frac{3}{2}d\right)^{\lambda+1}\right) < \sum_{k=1}^{n}(a+kd)^\lambda$$
$$< \frac{(a+d)^\lambda + (a+nd)^\lambda}{2} + \frac{1}{d(\lambda+1)}\left((a+nd)^{\lambda+1} - (a+d)^{\lambda+1}\right). \tag{36}$$

**Proof.** Taking $f(x) = x^\lambda$ $(x > 0)$ in Corollary 2. It is clear that $f$ is strictly convex and positive on $(0, +\infty)$ for $\lambda < 0$ or $\lambda > 1$, and $f$ is strictly concave and positive on $(0, +\infty)$ for $0 < \lambda < 1$. After some simple calculations, we deduce inequalities (35) and (36) from Corollary 2. □

The following examples provide some interesting applications of Proposition 5.

**Example 1.** Let $S = \sqrt[5]{1} + \sqrt[5]{2} + \sqrt[5]{3} + \cdots + \sqrt[5]{9999}$. Determine the integer part of $S$.

*Solution*. Putting $a = 0$, $d = 1$, $n = 9999$, $\lambda = \frac{1}{5}$ in (36), we obtain

$52576.2683 < S < 52576.2906$ which implies that $[S] = 52576$,

where $[S]$ denotes the integer part of $S$.

**Example 2.** Let $S = \frac{1}{\sqrt[7]{1}} + \frac{1}{\sqrt[7]{3}} + \frac{1}{\sqrt[7]{5}} + \cdots + \frac{1}{\sqrt[7]{9999}}$. Determine the integer part of $S$.

*Solution*. Since $S = 1 + \sum_{k=1}^{4999} 1/\sqrt[7]{1+2k}$, putting $a = 1$, $d = 2$, $n = 4999$, $\lambda = -\frac{1}{7}$ in (35) yields

$1564.8374 < S < 1564.8466$ which implies that $[S] = 1564$.

**Example 3.** Let $S = \sqrt[6]{2} + \sqrt[6]{4} + \sqrt[6]{6} + \cdots + \sqrt[6]{2006}$. Determine the integer part of $S$.

*Solution*. Putting $a = 0$, $d = 2$, $n = 1003$, $\lambda = \frac{1}{6}$ in (36), we get

$3054.4653 < S < 3054.4860$ which implies that $[S] = 3054$.

## Acknowledgment

The research was supported by the Natural Science Foundation of Fujian Province of China under grant No. S0650003.

## References

[1] S. Wu, H.N. Shi, Majorization proofs of inequalities for convex sequences, Math. Practice Theory 33 (12) (2003) 132–137.
[2] A.M. Mercer, Polynomials and convex sequence inequalities, J. Inequal. Pure Appl. Math. 6 (1) (2005) Art. 8.
[3] D.S. Mitrinović, P.M. Vasić, Analytic Inequalities, Springer-Verlag, New York, 1970.
[4] A.W. Roberts, D.E. Varberg, Convex Functions, Academic Press, New York, 1973.
[5] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.
[6] A.W. Marshall, I. Olkin, Inequalities: The Theory of Majorization and its Applicaitons, Academic Press, New York, 1979.
[7] S. Wu, Some results on extending and sharpening the Weierstrass product inequalities, J. Math. Anal. Appl. 308 (2) (2005) 689–702.
[8] S. Wu, Generalization and sharpness of power means inequality and their applications, J. Math. Anal. Appl. 312 (2) (2005) 637–652.
[9] G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, second ed., Cambridge Univ. Press, Cambridge, 1952.
[10] S. Wu, Schur-convexity for a class of symmetric functions and its applications, Math. Practice Theory 34 (12) (2004) 162–172.
[11] A.M. Mercer, A note on a paper by S. Haber, Internat. J. Math. Math. Sci. 6 (3) (1983) 609–611.
[12] S. Wu, The generalization of an inequality for convex sequence, J. Chengdu Univ. (Natural Science) 23 (3) (2004) 11–15.